Quadratic forms and eigenvalues 2

Given the following quadratic forms, we request:

1. Write each quadratic form in matrix form.
2. Compute the eigenvalues of the associated matrix.

d) In $\mathbb{R}^2$

$$\phi(x_1, x_2) = 4x_1^2 + 4\,x_1x_2 + 7x_2^2.$$

e) In $\mathbb{R}^2$

$$\phi(x_1, x_2) = x_1^2 + 2\,x_1x_2 + x_2^2.$$

f) In $\mathbb{R}^3$

$$\phi(x_1, x_2, x_3) = 2x_1^2 + 4\,x_1x_2 + 2x_2^2 - 3x_3^2.$$

g) In $\mathbb{R}^3$

$$\phi(x_1, x_2, x_3) = x_1^2 + 4x_2^2 + 9x_3^2 + 4\,x_1x_3.$$

Solution

d)

Let the quadratic form be

$$\phi(x_1, x_2) = 4x_1^2 + 4x_1x_2 + 7x_2^2.$$

1) Matrix form

To express ϕ in matrix form, we seek a symmetric matrix Q such that

$$\phi(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We observe that:

- The coefficient of x_1^2 is 4, so $q_{11} = 4$.
- The coefficient of x_2^2 is 7, so $q_{22} = 7$.
- The term $4x_1x_2$ is written as $2q_{12}x_1x_2$, from which $2q_{12} = 4$, so $q_{12} = 2$.

Thus, the associated matrix is

$$Q = \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix}.$$

Hence, we have:

$$\phi(x_1, x_2) = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 4 & 2 \\ 2 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

2) Eigenvalues of the associated matrix

To find the eigenvalues of Q, we solve

$$\det(Q - \lambda I) = 0.$$

Computing:

$$\det \begin{pmatrix} 4 - \lambda & 2 \\ 2 & 7 - \lambda \end{pmatrix} = (4 - \lambda)(7 - \lambda) - 2 \cdot 2.$$

Expanding:

$$(4 - \lambda)(7 - \lambda) = 28 - 11\lambda + \lambda^2,$$

then,

$$\det(Q - \lambda I) = \lambda^2 - 11\lambda + 28 - 4 = \lambda^2 - 11\lambda + 24.$$

Factoring the polynomial:

$$\lambda^2 - 11\lambda + 24 = (\lambda - 8)(\lambda - 3) = 0.$$

From which we obtain the eigenvalues:

$$\lambda_1 = 8 \quad \text{and} \quad \lambda_2 = 3.$$

e) In $\mathbb{R}^2$

Let the quadratic form be

$$\phi(x_1, x_2) = x_1^2 + 2\,x_1x_2 + x_2^2.$$

1) Matrix form

To express ϕ in matrix form, we seek a symmetric matrix Q such that

$$\phi(x_1, x_2) \;=\; \begin{pmatrix} x_1 & x_2 \end{pmatrix} Q \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We observe that:

- The coefficient of x_1^2 is 1, so $q_{11} = 1$.
- The coefficient of x_2^2 is 1, so $q_{22} = 1$.
- The term $2\,x_1x_2$ is written as $2q_{12}\,x_1x_2$, from which $2q_{12} = 2$, so $q_{12} = 1$.

Thus, the associated matrix is

$$Q \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Hence, we have:

$$\phi(x_1, x_2) \;=\; \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

2) Eigenvalues of the associated matrix

To find the eigenvalues of Q, we solve the equation

$$\det(Q - \lambda I) = 0.$$

We compute:

$$\det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} \;=\; (1-\lambda)^2 - 1.$$

Expanding the expression:

$$(1-\lambda)^2 - 1 \;=\; 1 - 2\lambda + \lambda^2 - 1 \;=\; \lambda^2 - 2\lambda.$$

Factoring:

$$\lambda^2 - 2\lambda \;=\; \lambda(\lambda - 2) = 0.$$

Thus, the eigenvalues are:

$$\lambda_1 = 0 \quad \text{and} \quad \lambda_2 = 2.$$

f) In $\mathbb{R}^3$

Let the quadratic form be

$$\phi(x_1, x_2, x_3) \;=\; 2x_1^2 \;+\; 4x_1x_2 \;+\; 2x_2^2 \;-\; 3x_3^2.$$

1) Matrix form

To express ϕ in matrix form, we seek a symmetric matrix Q such that

$$\phi(x_1, x_2, x_3) \;=\; \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} Q \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Comparing the coefficients, we have:

- The coefficient of x_1^2 is 2, so $q_{11} = 2$.
- The coefficient of x_2^2 is 2, so $q_{22} = 2$.
- The coefficient of x_3^2 is -3, so $q_{33} = -3$.
- The term $4x_1x_2$ corresponds to $2q_{12}\,x_1x_2$, hence $2q_{12} = 4$ and thus $q_{12} = 2$.
- There are no mixed terms involving x_3, so $q_{13} = q_{23} = 0$.

The associated matrix is therefore

$$Q = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}.$$

Thus, we have:

$$\phi(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

2) Eigenvalues of the associated matrix

The matrix Q is block-diagonal, as it can be written as

$$Q = \left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \\ \hline 0 & 0 & -3 \end{array}\right).$$

The 2×2 block is

$$\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

To find its eigenvalues, we solve

$$\det \begin{pmatrix} 2-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 4 = 0.$$

Expanding:

$$(2-\lambda)^2 - 4 = 4 - 4\lambda + \lambda^2 - 4 = \lambda^2 - 4\lambda.$$

Factoring:

$$\lambda(\lambda - 4) = 0.$$

Thus, the eigenvalues of the 2×2 block are $\lambda = 0$ and $\lambda = 4$.

The third eigenvalue corresponds to the 1×1 block and is:

$$\lambda = -3.$$

Thus, the eigenvalues of Q are:

$$\lambda_1 = 4, \quad \lambda_2 = 0, \quad \lambda_3 = -3.$$

g) In $\mathbb{R}^3$

Let the quadratic form be

$$\phi(x_1, x_2, x_3) \;=\; x_1^2 \;+\; 4x_2^2 \;+\; 9x_3^2 \;+\; 4\,x_1x_3.$$

1) Matrix form

To express ϕ in matrix form, we seek a symmetric matrix Q such that

$$\phi(x_1, x_2, x_3) \;=\; \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} Q \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

We observe that:

- The coefficient of x_1^2 is 1, so $q_{11} = 1$.
- The coefficient of x_2^2 is 4, so $q_{22} = 4$.
- The coefficient of x_3^2 is 9, so $q_{33} = 9$.
- The term $4x_1x_3$ is written as $2q_{13}x_1x_3$, from which $2q_{13} = 4$ and thus $q_{13} = 2$.
- There are no mixed terms involving x_1x_2 or x_2x_3, so $q_{12} = q_{23} = 0$.

The associated matrix is therefore

$$Q \;=\; \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 9 \end{pmatrix}.$$

Thus, we have:

$$\phi(x_1, x_2, x_3) \;=\; \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 4 & 0 \\ 2 & 0 & 9 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

2) Eigenvalues of the associated matrix

To find the eigenvalues of Q, we solve the characteristic equation

$$\det(Q - \lambda I) = 0.$$

Given that

$$Q - \lambda I = \begin{pmatrix} 1-\lambda & 0 & 2 \\ 0 & 4-\lambda & 0 \\ 2 & 0 & 9-\lambda \end{pmatrix},$$

we can compute the determinant using expansion along the second row (or notice that the matrix is nearly block-diagonal). Indeed, we have:

$$\det(Q - \lambda I) = (4 - \lambda) \cdot \det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 9-\lambda \end{pmatrix}.$$

We compute the determinant of the 2×2 block:

$$\det \begin{pmatrix} 1-\lambda & 2 \\ 2 & 9-\lambda \end{pmatrix} = (1-\lambda)(9-\lambda) - 2 \cdot 2.$$

Expanding,

$$(1-\lambda)(9-\lambda) = \lambda^2 - 10\lambda + 9,$$

so that

$$(1-\lambda)(9-\lambda) - 4 = \lambda^2 - 10\lambda + 9 - 4 = \lambda^2 - 10\lambda + 5.$$

Thus, the characteristic equation is:

$$(4-\lambda)(\lambda^2 - 10\lambda + 5) = 0.$$

From which we obtain the eigenvalues:

1. $\lambda = 4$ (from $4 - \lambda = 0$).
2. The other eigenvalues satisfy $\lambda^2 - 10\lambda + 5 = 0$. Using the quadratic formula:

$$\lambda = \frac{10 \pm \sqrt{100-20}}{2} = \frac{10 \pm \sqrt{80}}{2} = \frac{10 \pm 4\sqrt{5}}{2} = 5 \pm 2\sqrt{5}.$$

Therefore, the eigenvalues are:

$$\lambda_1 = 4, \quad \lambda_2 = 5 + 2\sqrt{5}, \quad \lambda_3 = 5 - 2\sqrt{5}.$$